Influence of Magnetic Reynolds Number on Power Generated by an Ideal MHD Device

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Nomenclature

A	= channel area
a	= electrode length
$\boldsymbol{B}$	= magnetic induction
$\boldsymbol{E}$	= electric field
h	= channel height
$\boldsymbol{J}$	= current density
K_y	= loading parameter
P_ℓ	= power delivered to load
Rm	= magnetic Reynolds number
$\boldsymbol{U}$	= velocity
w	= channel width
x_B	= magnetic field extent
μ_0	= permeability of free space
σ	= electrical conductivity

IN this Note a simple expression is derived for the power of an ideal, one-dimensional MHD generator that is not restricted by the usual assumption of a very small magnetic Reynolds number. It applies to pulsed MHD generators and possibly other devices. It shows that, as the magnetic Reynolds number increases, the power to the load approaches an asymptotic limit.

Consider MHD flow in a channel of constant cross section $A = wh$ with components of velocity $\boldsymbol{U} = (U_x,0,0)$, magnetic induction $\boldsymbol{B} = (0,0,B_z)$, and continuous electrodes of length a along the x axis on the y walls at $y=0$ and $y=\mathrm{h}$. There is no externally applied electric potential (e.g., through batteries) and the conductivity σ is scalar and uniform in a region extending from $x=0$ to a, $y=0$ to h, and $z=0$ to w. Let a be smaller than the extent of the magnetic field region x_B along the x axis, so that $a<x_B$ and the applied magnetic field B_0 is constant in the region a and zero elsewhere. The dimensionality convention used here for the MHD problem follows the convention established previously.[1]

The electric field induced in the plasma in this region due to the streaming of the plasma across $\boldsymbol{B}$ is $\boldsymbol{U}\times\boldsymbol{B}$ and the electric field in this region across the electrodes due to an external load is $\boldsymbol{E}$. Then, the resultant electric field in the plasma $\boldsymbol{E}'$ is given by

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{U}\times\boldsymbol{B} \tag{1}$$

This electric field causes a current of density $\boldsymbol{J}$ to flow in the plasma. This current interacts with the magnetic field $\boldsymbol{B}$ to produce a force on the plasma per unit volume given by $\boldsymbol{J}\times\boldsymbol{B}$.

Let the problem be idealized by considering the electric field vectors to be $\boldsymbol{E} = (E_y,0,0)$ and $\boldsymbol{E}' = (E_y',0,0)$ and let all parameters except B_z and J_y be constant and uniform. Let K_y be the loading parameter as usually defined, so that

$$E_y = K_y U_x B_0 \tag{2}$$

where B_0 is the applied magnetic induction. Note that E_y can be caused either by applying a load or by applying a power supply externally across the electrodes. Also note that B_z is the total local magnetic induction that includes both the applied and induced magnetic field components. At high magnetic Reynolds numbers, B_z varies with x. Figure 1 shows the orientation of all these vectors.

Currently favored expressions, such as Eq. (11) given below, for the power generated by a simple, one-dimensional, ideal generator are usually given for the case where the magnetic Reynolds number Rm is much smaller than unity ($Rm \ll 1$). These low Rm expressions make no assumption concerning the region of high conductivity in the x direction a. In fact, they assume that the magnetic field $\boldsymbol{B}$ is a square function (i.e., they assume $B_z \equiv B_0$ in $0 \le x \le a$ and zero elsewhere). Since for low Rm it is true that $B_z \simeq B_0$, this assumption is valid. For a continuously firing MHD generator (i.e., not pulsed), the extent of the high-conductivity region can be assumed to be infinite (i.e., σ is high in the region from $x=0$ to ∞ and only the extent of $\boldsymbol{B}$ is limited).

In this analysis, we consider the extent of the high-conductivity region to be limited to $0 \le x \le a$ and $a < x_B$.

From Ohm's law we obtain, for the given geometry,

$$J_y = \sigma E_y' \tag{3}$$

while from Eq. (1), E_y' is given by

$$E_y' = E_y - U_x B_z \tag{4}$$

The Maxwell equation governing B_z in terms of the current J_y is

$$-\frac{\partial B_z}{\partial x} = \mu_0 J_y \tag{5}$$

The boundary condition to be applied to Eq. (5) is that the induced magnetic field immediately in front of the plasma must exactly oppose the corresponding induced magnetic field at the trailing edge of the plasma or $\nabla\cdot\boldsymbol{B}=0$. If B^+

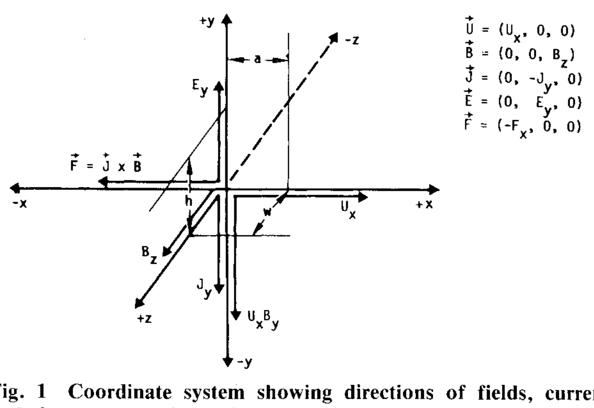

Fig. 1 Coordinate system showing directions of fields, currents, and forces as well as limits of integration for a simple, one-dimensional Faraday MHD generator without restrictions in magnetic Reynolds number. If one is positioned downstream of the exit of the MHD generator and looks upstream, and if the needle of a compass that currently (1982) points toward the north pole of the Earth points to the left, then the cathodes (emitting electrons into the plasma) of the generator will be on the bottom and the anodes on the top of the channel with respect to the observer. Conversely, if the generator is designed so that cathodes are on the bottom and anodes on the top, the north-pointing needle of the compass must point to the left of this observer, i.e., the *B* field goes from right to left to generate power.

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and B^- denote the fields at the forward and trailing edge of the plasma, this condition is

$$B^+ + B^- = 2B_0 \tag{6}$$

Combining Eqs. (2-5) so as to eliminate either B_z or J_y and solving subject to the boundary condition (6) then results in

$$B_z = B_0 K_y + \frac{2B_0(1-K_y)}{(1+e^{Rm})} e^{Rmx/a} \tag{7}$$

and

$$J_y = -\frac{2B_0 \sigma U_x (1-K_y) \cdot e^{Rmx/a}}{(1+e^{Rm})} \tag{8}$$

where

$$Rm = \mu_0 \sigma U_x a \tag{9}$$

is the magnetic Reynolds number for the given geometry.

The local power density supported by the plasma is $P = \boldsymbol{J} \cdot \boldsymbol{E} = J_y E_y$. The total power delivered to the load is the negative integral of this quantity over the region of the plasma. Using Eqs. (2) and (8) in this integral, we obtain

$$P_\ell / A = 4\left(\frac{B_0^2}{2\mu_0}\right) U_x K_y (1-K_y) \tanh\left(\frac{Rm}{2}\right) \tag{10}$$

This expression can be thought of as the equivalent to the low magnetic Reynolds number expression given by

$$(P_\ell / A)_{\text{low } Rm} = (B_0^2/\mu_0) U_x K_y (1-K_y) Rm \tag{11}$$

Equation (11) is familiar to many[2] and is used for order-of-magnitude calculations. Great care must be exercised in the use of Eq. (10) because of the approximations involved in its derivation. These approximations include the basic statements of the dimensionality of the MHD problem, i.e., the neglect of components other than $\boldsymbol{U} = (U_x,0,0)$, $\boldsymbol{B} = (0,0,B_z)$, $\boldsymbol{J} = (0,J_y,0)$, $\boldsymbol{E} = (0,E_y,0)$ for the various vector quantities, the form of the Maxwell equations used, and the uniformity of the various quantities or the parameters $\sigma, U_x, E_y, B_0, J_y, K_y$ as well as the sharpness of the boundary conditions.

In general, other vector components of the fields and the fluid dynamic parameters will exist and the quantities σ, U_x, B_0, K_y, E_y, and B_z will vary with all three coordinates, giving rise under some conditions to internal circulating eddy currents.[3] Thus, multidimensional computations are required for more detailed design analyses. Also, the conductivity is generally a tensor and the load may vary with x in the case of multielectrode generators and *all* the variables may vary with time, further adding to the complications.

Note that since $\tanh(Rm/2) \rightarrow 1$ as $Rm \rightarrow \infty$, the power obtained from a high magnetic Reynolds number device (e.g., one with very high conductivity) will reach the limit

$$P_\ell / A = 4(B_0^2/2\mu_0) U_x K_y (1-K_y) \tag{12}$$

and using the load-matching condition $K_y = ½$ we obtain for the maximum power from an MHD generator at high magnetic Reynolds numbers

$$(P_\ell / A)_{\max} = (B_0^2/2\mu_0) U_x \tag{13}$$

Note that, since already at $Rm = 3$, $\tanh(Rm/2) > 0.9$, for $U = 10^4$ m/s and $a = 1$ m, we obtain $\sigma = 239$ mho/m. Therefore, there is an optimum conductivity for each pulsed MHD generator given by $\sigma \simeq 3/(\mu_0 U a)$. This expression can be derived by reasoning that operation at a higher magnetic Reynolds number is not required, especially since other losses or costs can only increase as Rm is increased. A higher conductivity is not necessary.

Note that Eq. (13) can be derived intuitively in a couple of ways (these results date from 1961 ± 1):

1) The magnetic pressure acting on the plasma in a region of magnetic field B_0 is $B_0^2/2\mu_0$. This is the force per unit area F/A acting to contain the plasma in the limit of very high magnetic Reynolds number and, therefore,

$$F/A = (B_0^2/2\mu_0) \tag{14}$$

The power that is produced by this plasma if it moves at a velocity U_x is therefore

$$U_x \cdot F/A = P_\ell / A = (B_0^2/2\mu_0) U_x \tag{15}$$

2) The energy contained in a magnetic field per unit volume is $B_0^2/2\mu_0$. At high magnetic Reynolds number a plasma sweeps out the magnetic field. The power removed per unit time associated with a one-dimensional sweep-out of the magnetic field by a plasma from region a will be

$$\text{Power}/wh = P_\ell / A = (B_0^2/2\mu_0) U_x \tag{16}$$

It is now appropriate to note that Eqs. (10) and (11) both approach zero as $Rm \rightarrow 0$. In other words, the power to the load vanishes as $Rm \rightarrow 0$. All other expressions are similarly well-behaved.

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References

[1]Demetriades, S. T., "Magnetohydrodynamic Orbit Control for Satellites," *Electrical Engineering*, Vol. 79, Dec. 1960, pp. 987-995.

[2]Rosa, R. J., *Magnetohydrodynamic Energy Conversion*, McGraw-Hill Book Co., New York, 1968.

[3]Oliver, D. A., Swean, T. F., Markham, D. M., Maxwell, C. D., and Demetriades, S. T., "High Magnetic Reynolds Number and Strong Interaction Phenomena in MHD Channel Flow," *Proceedings of the Seventh International Conference on MHD Electrical Power Generation*, Massachusetts Institute of Technology, Cambridge, Vol. II, 1980, p. 565.

Freestream Turbulence Effects on Turbulent Boundary Layers in an Adverse Pressure Gradient

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Nomenclature

C = constant in the law of the wall

K = von Kármán constant

Tu = turbulence intensity, $Tu = \tilde{u}/U_\infty$

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